UCHBURCHAKLAR UCHUN AYRIM TEOREMALARNI KOORDINATALAR METODI YORDAMIDA ISBOTLASH

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Annotatsiya. Ayrim teoremalarni koordinatalar metodi yordanida isbotlash qulay va osonroqdir. Ihtiyoriy uchburchakning ortomarkazi, og'irlik markazi va tashqi chizilgan aylana markazi haqidagi Eyler teoremasining va Cheva teoremasining sintetik usuldagi isbotlari elementar geometriyaga oid adabiyotlarda bayon qilingan. Bu teoremaning analitik usuldagi isboti esa tegishli adabiyotlarda uchramaydi. Bu ishda bu teoremalar analitik usulda isbotlangan.

Kalit so'zlar: Cheva teoremasi, Eyler teoremasi, analitik metod, uchburchak.

Аннотация. Некоторые теоремы удобнее и проще доказывать методом координат. В учебных пособиях и учебниках теореме Чева и теорема Эйлера аналитическим методом не доказана. В этой работе теорема доказана аналитическим методом.

Ключевые слова: теореме Чева, теорема Эйлера, аналитический метод, треугольник

Abstract. Some theorems are more convenient and easier to prove by the method of coordinates. In textbooks and textbooks, Chew's and Eyler's theorems has not been proved analytically. In this paper, the theorems is proved analytically.

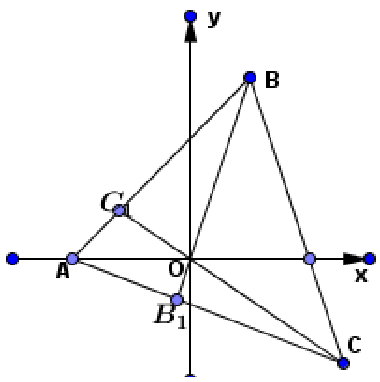
Keywords: Chew's theorem, Eyler's theorem, analytical method, triangle

Teorema (Cheva).[2,11-b.] Agar O nuqta ABC uchburchakning ichki sohasida yotuvchi ihtiyoriy nuqta bo'lsa, AO, BO, CO to'g'ri chiziqlar uchburchakning BC, CA, AB tomonlarini mos ravishda A_1, B_1, C_1 nuqtalarda kessa, u holda ushbu munosabat o'rinlidir:

$$\frac{AC_1}{C_1B}\cdot\frac{CB_1}{B_1A}\cdot\frac{BA_1}{A_1C}=1 \qquad (1)$$

Isbot: To'g'ri burchakli koordinatalar sistemasini shunday tanlaymizki, koordinatalar boshi O nuqtada, A, O, A_1 nuqtalar absissa o'qida yotsin (1-chizma).

Uchburchakning uchlari quyidagi koordinatalarga ega bo'lsin:



(1-chizma)

$A(a,0), B(b_1,b_2), C(c_1,c_2).$

Bu holda AO, BO, CO to'g'ri chiziqlari quyidagi tenglamalarga ega bo'ladi:
AO tenglamasi: $y = 0$.
CO tenglamasi: $\frac{x}{c_1} = \frac{y}{c_2}$ yoki $c_2x - c_1y = 0$.
BO tenglamasi: $b_2x - b_1y = 0$.
ABC uchburchak tomonlarining tenglamalari quyidagicha:
AB tomon tenglamasi $\frac{x-a}{b_1-a} = \frac{y}{b_2}$ yoki $b_2x - (b_1 - a)y - ab_2 = 0$.
BC tomon tenglamasi $(c_2 - b_2)x - (c_1 - b_1)y - b_1(c_2 - b_2) + b_2(c_1 - b_1) = 0$. (1-chizma)
AC tomon esa $c_2x - (c_1 - a)y - ac_2 = 0$ tenglamaga ega.

Endi A_1, B_1, C_1 nuqtalarning koordinatalarini aniqlaymiz. A_1 nuqta BC va AO to'g'ri chiziqlarning kesishuv nuqtasi. Shuning uchun uning koordinatalari

$$\begin{cases} (c_2 - b_2)x - (c_1 - b_1)y - b_1(c_2 - b_2) + b_2(c_1 - b_1) = 0 \\ y = 0 \end{cases}$$

tenglamalar sistemasining yechimlaridan iborat.
Bu sistemani yechib, A_1 nuqta koordinatalari

$$A_1\left(\frac{b_1(c_2 - b_2) - b_2(c_1 - b_1)}{c_2 - b_2}, 0\right)$$

ko'rinishda ekanligini ko'ramiz. Shunga o'hshash

$$\begin{cases} b_2x - b_1y = 0 \\ c_2x - (c_1 - a)y - ac_2 = 0, \end{cases} \quad \begin{cases} b_2x - (b_1 - a)y - ab_2 = 0 \\ c_2x - c_1y = 0 \end{cases}$$

tenglamalar sistemasini yechib, B_1, C_1 nuqtalarning koordinatalarini aniqlaymiz:

$$B_1\left(\frac{ab_1c_2}{b_1c_2 - (c_1 - a)b_2}, \frac{ab_2c_2}{b_1c_2 - (c_1 - a)b_2}\right), \; C_1\left(\frac{ac_1b_2}{c_1b_2 - (b_1 - a)c_2}, \frac{ab_2c_2}{c_1b_2 - (b_1 - a)c_2}\right).$$

Endi

$$\overrightarrow{AC_1} = \lambda_1 \overrightarrow{C_1B}, \; \overrightarrow{BA_1} = \lambda_2 \overrightarrow{A_1C}, \quad \overrightarrow{CB_1} = \lambda_3 \overrightarrow{B_1A}$$

tengliklarni qanoatlantiruvchi

$$\lambda_1 = \frac{AC_1}{C_1B}, \; \lambda_2 = \frac{BA_1}{A_1C}, \quad \lambda_3 = \frac{CB_1}{B_1A}$$

sonlarni aniqlaymiz.
Buning uchun $\overrightarrow{AC_1}, \overrightarrow{C_1B}, \overrightarrow{BA_1}, \overrightarrow{A_1C}, \overrightarrow{CB_1}, \overrightarrow{B_1A}$ vektorlarning koordinatalarini aniqlaymiz.

$$AC_1\left(\frac{ac_1b_2}{c_1b_2 - (b_1 - a)c_2} - a, \frac{ab_2c_2}{c_1b_2 - (b_1 - a)c_2} - 0\right) = AC_1\left(\frac{ac_2(b_1 - a)}{c_1b_2 - (b_1 - a)c_2}, \frac{ab_2c_2}{c_1b_2 - (b_1 - a)c_2}\right),$$

$$C_1B\left(\frac{(b_1 - a)(c_1b_2 - b_1c_2)}{c_1b_2 - (b_1 - a)c_2}, \frac{b_2(c_1b_2 - b_1c_2)}{c_1b_2 - (b_1 - a)c_2}\right).$$

$\overrightarrow{AC_1} = \lambda_1 \overrightarrow{C_1B}$ tenglikni qanoatlantiruvchi λ_1 sonini

$$\frac{ac_2(b_1-a)}{c_1b_2-(b_1-a)c_2}=\lambda_1\frac{(b_1-a)(c_1b_2-b_1c_2)}{c_1b_2-(b_1-a)c_2},\quad \frac{ab_2c_2}{c_1b_2-(b_1-a)c_2}=\lambda_1\frac{b_2(c_1b_2-b_1c_2)}{c_1b_2-(b_1-a)c_2}.$$

tengsizlikni qanoatlantiradigan qilib olamiz. Oxirgi sistemani

$$\lambda_1=\frac{ac_2}{c_1b_2-b_1c_2} \qquad (2)$$

qiymat qanoatlantiradi.

$\overrightarrow{BA_1}=\lambda_2\overrightarrow{A_1C}$ tenglikni qanoatlantiruvchi λ_2 sonini aniqlash uchun $\overrightarrow{BA_1}$ va $\overrightarrow{A_1C}$ vektor koordinatalaridan foydalanamiz. Yuqoridagiga o'hshash bu vektorlar uchun

$$\overrightarrow{BA_1}\left(\frac{b_2(b_1-c_1)}{c_2-b_2},-b_2\right),\ \overrightarrow{A_1C}\left(\frac{c_2(c_1-b_1)}{c_2-b_2},c_2\right)$$

koordinatalarni olish qiyin emas. λ_2 sonini

$$\frac{b_2(b_1-c_1)}{c_2-b_2}=\lambda_2\frac{c_2(c_1-b_1)}{c_2-b_2},\ -b_2=\lambda_2c_2$$

tengliklardan topamiz:

$$\lambda_2=-\frac{b_2}{c_2}. \qquad (3)$$

$\overrightarrow{CB_1}=\lambda_3\overrightarrow{B_1A}$ tenglikdan λ_3 sonini aniqlaymiz. Vektor koordinatalari quyidagicha:

$$\overrightarrow{CB_1}\left(\frac{(c_1-a)(c_1b_2-b_1c_2)}{b_1c_2-(c_1-a)b_2},\frac{c_2(c_1b_2-b_1c_2)}{b_1c_2-(c_1-a)b_2}\right),\ \overrightarrow{B_1A}\left(\frac{ab_2(a-c_1)}{b_1c_2-(c_1-a)b_2},\frac{-ab_2c_2)}{b_1c_2-(c_1-a)b_2}\right).$$

$\overrightarrow{BA_1}$ vektor koordinatalarini λ_3 ga ko'paytirib, $\overrightarrow{CB_1}$ vektorninng mos koordinatalariga tenglashtirsak, hosil bo'lgan tengliklardan

$$\lambda_3=-\frac{c_1b_2-b_1c_2}{ab_2} \qquad (4)$$

qiymatni aniqlaymiz.

Yuqorida $\lambda_1,\lambda_2,\lambda_3$ sonlari uchun olingan (2), (3), (4) qiymatlarni ko'paytiramiz:

$$\frac{AC_1}{C_1B}\cdot\frac{BA_1}{A_1C}\cdot\frac{CB_1}{B_1A}=\lambda_1\cdot\lambda_2\cdot\lambda_3=\frac{ac_2}{c_1b_2-b_1c_2}\cdot\left(-\frac{b_2}{c_2}\right)\cdot\left(-\frac{c_1b_2-b_1c_2}{ab_2}\right)=1$$

Cheva teoremasi to'liq isbotlandi.

Yuqoridagi keltirilgan usuldan foydalanib, Chevaning quyidagi teskari teoremasini ham isbotlash qiyin emas:

Chevaning teskari teoremasi.[2, 12-b.] Agar ABC uchburchakning AB, BC, CA tomonlarida mos ravishda olingan C_1, A_1, B_1 nuqtalar uchun (1) tenglik bajarilsa, u holda AA_1, BB_1, CC_1 to'g'ri chiziqlar o'zaro bir nuqtada kesishadi.

Isbot. C_1 nuqta AB tomonni $\lambda_1=\dfrac{ac_2}{c_1b_2-b_1c_2}$ nisbatda bo'ladi. Bundan foydalanib C_1 nuqta koordinatalari aniqlanadi:

$$C_1\left(\frac{ac_1b_2}{c_1b_2-b_1c_2+ac_2},\frac{ab_2c_2}{c_1b_2-b_1c_2+ac_2}\right).$$

CC_1 to‘g‘ri chiziq tenglamasi tuzilsa bu tenglama $Fx+Fy=0$ ko‘rinishda bo‘lib, $O(0,0)$ nuqta koordinatlarini qanoatlantiradi.

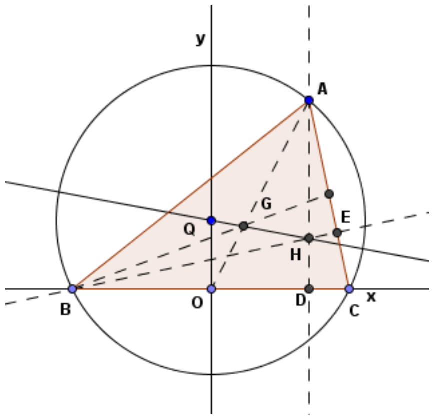
Ikkinchi tomondan, $O(0,0)$ nuqta BB_1 va AA_1 to‘g‘ri chiziqlarning kesishuv nuqtasidir.

Teorema isbotlandi.

Teorema (Eyler). Ihtiyoriy ABC uchburchakning ortomarkazi, og‘irlik markazi va shu uchburchakka tashqi chizilgan aylana markazi bir to‘g‘ri chiziqda yotadi.

Isbot: To‘g‘ri burchakli koordinatalar sistemasini shunday tanlaymizki, berilgan ABC uchburchakning BC tomoni absissa o‘qida yotib, koordinatalar boshi O nuqta BC tomon o‘rtasida yotsin (2-chizma). Tashqi chizilgan aylana markazini Q,ortomarkazni H, og‘irlik markazini G,orqali belgilaymiz. Koordinatalar sistemasini tanlanishiga ko‘ra G nuqta Oy ordinate o‘qida yotadi.

ABC uchburchak uchlari va unga tegishli nuqtalar quyidagi koordinatalarga ega bo‘lsin:



(2-chizma)

$A(a_1,a_2),\ B(-b,0),\ C(b,0),\ Q(0,q),\ G(x_1,y_1),\ H(x_2,y_2),\ AQ=BQ=CQ$ bo‘lib, $AQ=\sqrt{a_1^2+(q-a_2)^2}=\sqrt{b^2+q^2}$ tenglikdan

$$a_1^2+q^2-2a_2q+a_2^2=b^2+q^2$$

yoki

$$q=\frac{a_1^2+a_2^2-b^2}{2a_2}.$$

Demak, Q nuqta $Q(0,\frac{a_1^2+a_2^2-b^2}{2a_2})$ koordinatalarga ega. $AD\|Oy$ bo‘lib, AD balandlik tenglamasi $x-a_1=0$ ko‘rinishga ega. $BE\perp AC$ bo‘lib, BE balandlik tenglamasini B nuqtadan o‘tib AC ga perpendikulyar to‘g‘ri chiziq sifatida tuzamiz. AC tomon tenglamasi:

$$\frac{x-a_1}{b-a_1}=\frac{y-a_2}{-a_2}$$

yoki

$$a_2x+(b-a_1)y-a_2b=0.$$

AC to‘g‘ri chiziqning normal vektori $\vec{n}_1(a_2,(b-a_1))$ koordinatalarga ega. Shuning uchun BE balandlik tenglamasini $(b-a_1)x-a_2y+c=0$ ko‘rinishda izlaymiz. Bu to‘g‘ri chiziq B

nuqtadan o'tganligi uchun $(b-a_1)(-b)-a_2\cdot 0+c=0$ yoki $c=b(b-a_1)$. Demak, BE to'g'ri chiziq tenglamasi

$$(BE):(b-a_1)x-a_2y+b(b-a_1)=0$$

ko'rinishga ega bo'ladi. Uchburchakning H ortomarkazi koordinatalarini

$$\begin{cases} x-a_1=0 \\ (b-a_1)x-a_2y+b(b-a_1)=0 \end{cases}$$

tenglamalar sistemasining yechimi sifatida aniqlaymiz:

$$y=\frac{(b-a_1)(b+a_1)}{a_2}=\frac{(b^2-a_1^2)}{a_2},\ H\left(a_1,\frac{(b^2-a_1^2)}{a_2}\right).$$

ABC uchburchak uchlarining koordinatalari bo'yicha uning og'irlik markazi G ning koordinatalarini aniqlaymiz:

$$G\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)=G\left(\frac{a_1+b-b}{3},\frac{a_2+0+0}{3}\right)=G\left(\frac{a_1}{3},\frac{a_2}{3}\right)$$

yani $G\left(\frac{a_1}{3},\frac{a_2}{3}\right)$ koordinatalarga ega. Shunday qilib, Q,H,G nuqtalar uchun

$$Q(0,\frac{a_1^2+a_2^2-b^2}{2a_2}),\ H(a_1,\frac{b^2-a_1^2}{a_2}),G\left(\frac{a_1}{3},\frac{a_2}{3}\right) \qquad (5)$$

koordinatalarga ega bo'lamiz. Shu Q,H,G nuqtalar bir to'g'ri chiziqda yotishini isbotlash talab etiladi. Buni ikki usulda isbotlash mumkin.

1-usul. $\overline{QH}$ va $\overline{QG}$ vektorlari kollinear bo'lishi uchun ularning mos koordinatalari proporsional bo'lishi zarur va yetarlidir. Shuning uchun bu vektorlarning koordinatalarini (5) dan foydalanib aniqlaymiz:

$$\overline{QH}\left(a_1-0,\frac{b^2-a_1^2}{a_2}-\frac{a_1^2+a_2^2-b^2}{2a_2}\right)=\overline{QH}\left(a_1,\frac{3b^2-3a_1^2-a_2^2}{2a_2}\right).$$

$$\overline{QG}\left(\frac{a_1}{3},\frac{a_2}{3}-\frac{a_1^2+a_2^2-b^2}{2a_2}\right)=\overline{QG}\left(\frac{a_1}{3},\frac{3b^2-3a_1^2-a_2^2}{6a_2}\right).$$

$\overline{QH}\parallel\overline{QG}$ bo'lishi uchun

$$\frac{a_1}{\frac{a_1}{3}}=3,\ \frac{3b^2-3a_1^2-a_2^2}{2a_2}:\frac{3b^2-3a_1^2-a_2^2}{6a_2}=3$$

proporsionallik o'rinli bo'ladi.

Demak, Q,H,G nuqtalar bir to'g'ri chiziqda yotadi.

2-usul. Q va G nuqtalardan o'tuvchi $Q\,G$ to'g'ri chiziq tenglamasini aniqlab, H nuqta koordinatalari bu tenglamani qanoatlantirishini isbotlash yetarlidir.

Ikki nuqtadan o'tuvchi to'g'ri chuzuq tenglamasiga asosan Q va G nuqtalardan o'tuvchi to'g'ri chiziq tenglamasini topamiz:

$$\frac{x-0}{\frac{a_1}{3}-0}=\frac{y-\frac{a_1^2+a_2^2-b^2}{2a_2}}{\frac{a_2}{3}-\frac{a_1^2+a_2^2-b^2}{2a_2}}$$

yoki

$$(3b^2-3a_1^2-a_2^2)x-2a_1a_2y+a_1^3+a_1a_2^2-a_1b^2=0. \qquad (6)$$

Bu tenglama QG to'g'ri chiziq tenglamasi bo'lib, H nuqta koordinatalari bu tenglamani qanoatlantirishini tekshiramiz:

$$(3b^2-3a_1^2-a_2^2)a_1-2a_1a_2\frac{b^2-a_1^2}{a_2}+a_1^3+a_1a_2^2-a_1b^2=$$

$$=3a_1b^2-3a_1^3-a_1a_2^2-2a_1b^2+2a_1^3+a_1^3+a_1a_2^2-a_1b^2=$$

$$=3a_1b^2-3a_1b^2-3a_1^3+3a_1^3-a_1a_2^2+a_1a_2^2=0$$

Demak H nuqta QG to'g'ri chiziqda yotadi.

Eyler teoremasi to'la isbotlandi.

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