MINKOWSKI DIFFERENCE OF n-DIMENSIONAL CUBES

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Abstract: In this work, the operation of Minkowski's difference on sets, which is used in many areas of mathematics, is covered. An exact formula for calculating the Minkowski difference of two cubes given in an n-dimensional space and having an arbitrary mutual location has been derived. The obtained result is a generalization of the author's results obtained in the calculation of Minkowski differences of squares in two-dimensional space and cubes in three-dimensional space.

Keywords: Minkowski sum, Minkowski difference, hyperplane, parallel transfer, orthogonal projection

The Minkowski difference is one of the operations related to the nature of elements of sets. Properties of the Minkowski difference and many results on finding it are given in works [1-4]. The concept of the Minkowski difference of sets is used in robotics, automation, differential games, optimizing the quality of electric lighting, and many other fields of engineering [5-6].

Continuing the considerations for finding the Minkowski difference of squares presented in [1] and generalizing the method presented in [7], we obtained sufficient and necessary conditions for the existence of the Minkowski difference of two n-dimensional cubes given in Euclidean space R^n. In order for the definition of an n-dimensional cube in the Euclidean space R^n to be one-valued, it is enough to give its $n+1$ vertices that do not lie in the same hyperplane and are located on edges from one vertex. Let C^A and C^B be cubes given by vertices A_0, A_1, A_2 ,..., A_n and $B_0, B_1, B_2,..., B_n$ respectively. We introduce the following notations:

$$\overrightarrow{A_0A_1} = \vec{a}_1, \overrightarrow{A_0A_2} = \vec{a}_2 ,..., \overrightarrow{A_0A_n} = \vec{a}_n; \qquad (1)$$

$$\overrightarrow{B_0B_1} = \vec{b}_1, \overrightarrow{B_0B_2} = \vec{b}_2 ,..., \overrightarrow{B_0B_3} = \vec{b}_3. \qquad (2)$$

The number of all diagonals of cube C^B is found by the expression 2^{n-1} and we can express the vectors corresponding to these diagonals by all combinations of vectors $\vec{b}_1, \vec{b}_2, ...\vec{b}_n$.

For example, when $n = 2$ is a two-dimensional cube, that is, a square, the number of diagonals is 2, but the number of the vectors corresponding to the diagonals is 4, and we express these vectors by all combinations of the two vectors $\vec{b}_1, \vec{b}_2$ that define the square and correspond to its sides:

$$\vec{d}_1 = \vec{b}_1 + \vec{b}_2,$$
$$\vec{d}_2 = -\vec{b}_1 + \vec{b}_2,$$
$$\vec{d}_3 = \vec{b}_1 - \vec{b}_2,$$
$$\vec{d}_4 = -\vec{b}_1 - \vec{b}_2.$$

Since $\left|\vec{d}_1\right| = \left|\vec{d}_4\right|$ and $\left|\vec{d}_2\right| = \left|\vec{d}_3\right|$, the vectors $\vec{d}_1$ and $\vec{d}_4$ represent one diagonal and the vectors $\vec{d}_2$ and $\vec{d}_3$ represent another diagonal.

When $n=3$ is a three-dimensional cube, the number of diagonals is 4, but the number of the vectors corresponding to the diagonals is 8, and we express these vectors by all combinations of the three vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$ that define the cube and correspond to its edges from one vertex:

$$\begin{aligned}
\vec{d}_1 &= \vec{b}_1 + \vec{b}_2 + \vec{b}_3, & \vec{d}_5 &= -\vec{b}_1 - \vec{b}_2 - \vec{b}_3,\\
\vec{d}_2 &= -\vec{b}_1 + \vec{b}_2 + \vec{b}_3, & \vec{d}_6 &= \vec{b}_1 - \vec{b}_2 - \vec{b}_3,\\
\vec{d}_3 &= \vec{b}_1 - \vec{b}_2 + \vec{b}_3, & \vec{d}_7 &= -\vec{b}_1 + \vec{b}_2 - \vec{b}_3,\\
\vec{d}_4 &= \vec{b}_1 + \vec{b}_2 - \vec{b}_3, & \vec{d}_8 &= -\vec{b}_1 - \vec{b}_2 + \vec{b}_3.
\end{aligned}$$

Here two vectors represent one diagonal.

When $n=4$ is a four-dimensional cube, a tesseract the number of diagonals is 8, but the number of the vectors corresponding to the diagonals is 16, and we express these vectors by all combinations of the four vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4$ that define the cube and correspond to its edges from one vertex:

$$\begin{aligned}
\vec{d}_1 &= \vec{b}_1 + \vec{b}_2 + \vec{b}_3 + \vec{b}_4, & \vec{d}_9 &= -\vec{b}_1 - \vec{b}_2 - \vec{b}_3 - \vec{b}_4,\\
\vec{d}_2 &= -\vec{b}_1 + \vec{b}_2 + \vec{b}_3 + \vec{b}_4, & \vec{d}_{10} &= \vec{b}_1 - \vec{b}_2 - \vec{b}_3 - \vec{b}_4,\\
\vec{d}_3 &= \vec{b}_1 - \vec{b}_2 + \vec{b}_3 + \vec{b}_4, & \vec{d}_{11} &= -\vec{b}_1 + \vec{b}_2 - \vec{b}_3 - \vec{b}_4,\\
\vec{d}_4 &= \vec{b}_1 + \vec{b}_2 - \vec{b}_3 + \vec{b}_4, & \vec{d}_{12} &= -\vec{b}_1 - \vec{b}_2 + \vec{b}_3 - \vec{b}_4,\\
\vec{d}_5 &= \vec{b}_1 + \vec{b}_2 + \vec{b}_3 - \vec{b}_4, & \vec{d}_{13} &= -\vec{b}_1 - \vec{b}_2 - \vec{b}_3 + \vec{b}_4,\\
\vec{d}_6 &= -\vec{b}_1 - \vec{b}_2 + \vec{b}_3 + \vec{b}_4, & \vec{d}_{14} &= \vec{b}_1 + \vec{b}_2 - \vec{b}_3 - \vec{b}_4,\\
\vec{d}_7 &= -\vec{b}_1 + \vec{b}_2 - \vec{b}_3 + \vec{b}_4, & \vec{d}_{15} &= \vec{b}_1 - \vec{b}_2 + \vec{b}_3 - \vec{b}_4,\\
\vec{d}_8 &= -\vec{b}_1 + \vec{b}_2 + \vec{b}_3 - \vec{b}_4, & \vec{d}_{16} &= \vec{b}_1 - \vec{b}_2 - \vec{b}_3 + \vec{b}_4.
\end{aligned}$$

Here, too, one diagonal is represented by two vectors. Similarly, the number of diagonals of an n-dimensional cube is 2^{n-1}, but the number of the vectors corresponding to the diagonals is 2^n and we can express these vectors by all combinations of vectors $\vec{b}_1, \vec{b}_2, ..., \vec{b}_n$:

$$\begin{aligned}
&\vec{d}_1 = \vec{b}_1 + \vec{b}_2 + \ldots + \vec{b}_n, \\
&\vec{d}_2 = -\vec{b}_1 + \vec{b}_2 + \ldots + \vec{b}_n, \\
&\vec{d}_3 = \vec{b}_1 - \vec{b}_2 + \ldots + \vec{b}_n, \\
&\ldots \\
&\vec{d}_{2^n} = -\vec{b}_1 - \vec{b}_2 - \ldots - \vec{b}_n.
\end{aligned} \tag{3}$$

Theorem. In order for the Minkowski difference $C^A \overset{*}{-} C^B$ not to be empty, it is necessary and sufficient that the length of the orthogonal projection of the vectors $\vec{d}_i,\ i=\overline{1,2^{n-1}}$ corresponding to all diagonals of the cube C^B to the vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$, should not be greater than the length of the vector $\vec{a}_1$.

Proof. There can be two cases when calculating the difference $C^A \overset{*}{-} C^B$.

In the first case, all $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ vectors are parallel to all $\vec{b}_1, \vec{b}_2, ..., \vec{b}_n$ vectors, respectively, then the orthogonal projections of vectors $\vec{d}_i,\ i=\overline{1,2^n}$ to vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ are equal to the vector length $\vec{b}_1$ that is,

$$\left|proj_{\vec{a}_j}\vec{d}_i\right| = \left|\vec{b}_1\right|,\ i=\overline{1,2^n},\ j=\overline{1,n}. \tag{4}$$

According to the determination of the Minkowski difference, in this case, in order to be able to place the cube C^B inside the cube C^A that is, so that the difference $C^A \overset{*}{-} C^B$ is not empty, the relation $\left|\vec{a}_1\right| \ge \left|\vec{b}_1\right|$ is necessary and sufficient. This means that $\left|proj_{\vec{a}_j}\vec{d}_i\right| \le \left|\vec{a}_1\right|,\ i=\overline{1,2^n},\ j=\overline{1,n}$.

In the second case, at least one of the vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ is not parallel to one of the corresponding vectors $\vec{b}_1, \vec{b}_2, ..., \vec{b}_n$. We assume that none of these vectors are parallel to each other in the general case. Then the lengths of orthogonal projections of vectors $\vec{d}_i,\ i=\overline{1,2^n}$ to vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ are found using the formula

$$\left|proj_{\vec{a}_j}\vec{d}_i\right| = \frac{\left|\left\langle \vec{a}_j, \vec{d}_i \right\rangle\right|}{\left|\vec{a}_j\right|},\ i=\overline{1,2^n},\ j=\overline{1,n}. \tag{5}$$

Here $\left\langle \vec{a}_j, \vec{d}_i \right\rangle$ is the scalar product of vectors $\vec{a}_j$ and $\vec{d}_i$. We designate the vectors whose length is the longest among the orthogonal projections of vectors $\vec{d}_i,\ i=\overline{1,2^n}$ onto vectors $\vec{a}_j$ and whose direction is the same as the direction of the vectors $\vec{a}_j$, respectively, as $\vec{b}_1^{'}, \vec{b}_2^{'}, ..., \vec{b}_n^{'}$. We construct an n-dimensional rectangular parallelepiped $P^{'}$ whose edges consist of vectors $\vec{b}_j^{'}$,

$j=\overline{1,n}$ and which contains the cube C^B. According to the construction of this rectangular parallelepiped, $P^{'} \overset{*}{-} C^B \neq \varnothing$ is valid. As in the first case, so that the parallelepiped $P^{'}$ can be placed inside the cube C^A by parallel displacement. It is necessary and sufficient that the edges of $P^{'}$ are not greater than the corresponding edges of C^A, i.e.

$$|\vec{a}_1| \geq |\vec{b}_1^{'}|,\ |\vec{a}_2| \geq |\vec{b}_2^{'}|, \ldots,\ |\vec{a}_n| \geq |\vec{b}_n^{'}| \tag{6}$$

from the relations (6),

$$|\vec{a}_1| \geq \left|proj_{\vec{a}_j} \vec{d}_i\right|,\ i=\overline{1,2^n}\ ,\ j=\overline{1,n}; \tag{7}$$

The theorem has been proved.

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