QO'SH TARTIBLI HILFER DIFFERENSIAL OPERATOR QATNASHGAN YUQORI TARTIBLI TENGLAMA UCHUN KOSHI MASALASI

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So'nggi o'n yilliklar davomida kars tartibli differensial tenglamalar o'zining amaliyotga tatbiqlari va modellashtirishdagi qulayligi tufayli matematikaning muhim tarmoqlaridan biriga aylanib bormoqda [1]-[2]. Boshqa tomondan esa kasr tartibli differensial operatorlarning ko'pgina ta'riflari va turlari taklif etilmoqda hamda ularning amaliy masalalarga tatbiqlari o'rganilmoqda. Xususan, Riman-Liuvil, Kaputo va Hilfer operatorlari shular jumlasidan.

Quyida biz ham Hilfer kasr tartibli operatorining umumlashmasi hisoblangan qo'sh tartibli Hilfer kasr tartibli differensial operatori qatnashgan quyidagi kasr tartibli differensial tenglamaning yechilish usullariga e'tibor qaratamiz:

$$D_{0+}^{(\alpha,\beta)\mu}u(t)-\lambda u(t)=f(t), \qquad (1)$$

bu yerda $n-1<\alpha,\ \beta\le n,\ n\in\mathbb{N}$, $0\le\mu\le1$, $D_{0+}^{(\alpha,\beta)\mu}$ esa qo'sh tartibli Hilfer kasr tartibli differensial operatori, ya'ni

$$D_{0+}^{(\alpha,\beta)\mu}y(t)=I_{0+}^{\mu(n-\alpha)}\left(\frac{d}{dt}\right)^{n}I_{0+}^{(1-\mu)(n-\beta)}y(t), \qquad (2)$$

bu yerda

$$I_{0+}^{\gamma}y(t)=\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}y(s)ds,\quad \gamma>0,$$

Riman-Liuvil kasr tartibli integral operatori.

Izoh. (2) ko'rinishdagi qo'sh tartibli Hilfer kasr tartibli differensial operatorini quyidagi ko'rinishda ham yozish mumkin:

$$D_{0+}^{(\alpha,\beta)\mu}y(t)=I_{0+}^{\gamma-\delta}D_{0+}^{\gamma}y(t),$$

bu yerda $\gamma=\beta+\mu(n-\beta)$, $\delta=\beta+\mu(\alpha-\beta)$, D_{0+}^{γ} -Riman-Liuvil kasr tartibli differensial operatori.

Masala B. (1) tenglamaning shunday yechimi topilsinki, u quyidagi boshlang'ich shartlarni

$$\lim_{t\to0+}\left(\frac{d}{dt}\right)^{k}I_{0+}^{(1-\mu)(n-\beta)}u(t)=\psi_k,\quad k=\overline{0,n-1}, \qquad (3)$$

qanoatlantirsin. Bu yerda $f(t)$ berilgan funksiya.

Teorema. Agar $f(t)$ funksiya $L^1(0,+\infty)$ ga tegishli integrallanuvchi funksiya bo'lsa, (1), (3) masalaning yechimi quyidagi ko'rinishda yoziladi:

$$u(t)=\sum_{k=0}^{n-1}\psi_k t^{\gamma-n+k}E_{\delta,\gamma-n+k+1}\left(\lambda t^{\delta}\right)+\int_0^t (t-\tau)^{\delta-1}E_{\delta,\delta}\left[\lambda(t-\tau)^{\delta}\right]f(\tau)d\tau,$$

bu yerda $\gamma=\beta+\mu(n-\beta)$, $\delta=\beta+\mu(\alpha-\beta)$.

Isbot. Qo'sh tartibli (2) ko'rinishdagi Hilfer operatori uchun Laplas almashtirishi quyidagi ko'rinishda aniqlanadi:

$$L\left\{D_{0+}^{(\alpha,\beta)\mu}u(t)\right\}=s^{\beta+\mu(\alpha-\beta)}L\{u(t)\}-s^{\mu(\alpha-n)}\sum_{k=0}^{n-1}s^{n-k-1}\left(\frac{d}{dt}\right)^k I_{0+}^{(1-\mu)(n-\beta)}u(0+). \quad (4)$$

(1) tenglamaning har ikkala tomoniga Laplas almashtirishini tatbiq etib quyidagiga ega bo'lamiz:

$$L\left\{D_{0+}^{(\alpha,\beta)\mu}u(t)\right\}-\lambda L\{u(t)\}=L\{f(t)\}.$$

(4) ifoda va (3) boshlang'ich shartlarga asosan ushbuni hosil qilamiz

$$L\{u\}=\frac{s^{\mu(\alpha-n)}\sum_{k=0}^{n-1}\psi_k s^{n-k-1}+L\{f\}}{s^{\beta+\mu(\alpha-\beta)}-\lambda}, \quad (5)$$

bu yerda $L\{u\}$ va $L\{f\}$ mos ravishda u va f funksiyalarning Laplas almashtirishlari.

Hozir biz Mittag-Lefler funksiyasi uchun Laplas almashtirishini ixtiyoriy $\alpha>0$, $\beta>0$, $\lambda\in\mathbb{C}$ keltiramiz, ya'ni

$$L\left\{t^{\beta-1}E_{\alpha,\beta}\left(\lambda t^{\alpha}\right)\right\}=\frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}, \quad \left(\mathrm{Re}(s)>|\lambda|^{1/\alpha}\right).$$

Laplas konvolyutsiyasi va yuqoridagi munosabatni hisobga olsak, quyidagi ifodalarga ega bo'lamiz:

$$L^{-1}\left\{\frac{s^{\mu(\alpha-n)+n-k-1}}{s^{\beta+\mu(\alpha-\beta)}-\lambda}\right\}=t^{\beta+\mu(n-\beta)-n+k}E_{\beta+\mu(\alpha-\beta),\,\beta+\mu(n-\beta)-n+k+1}\left(\lambda t^{\beta+\mu(\alpha-\beta)}\right),$$

$$L^{-1}\left\{\frac{L\{f\}}{s^{\beta+\mu(\alpha-\beta)}-\lambda}\right\}=\int_0^t (t-\tau)^{\beta+\mu(\alpha-\beta)-1}E_{\beta+\mu(\alpha-\beta),\,\beta+\mu(\alpha-\beta)}\left[\lambda(t-\tau)^{\beta+\mu(\alpha-\beta)}\right]f(\tau)d\tau,$$

bu yerda L^{-1} teskari Laplas almashtirishi.

Ushbu ikkita munosabatni hisobga olib, (5) tenglikka teskari Laplas almashtirishini tatbiq etsak, (1), (3) masalaning teoremada keltirilgan yechimini hosil qilamiz. Teorema isbotlandi.

Endi ushbu (1), (3) masalani integral tenglamaga keltirish orqali yechishni ko'rsataylik. Bunda bizga quyidagi lemma yordamga keladi.

Lemma. Agar $f(t)\in L^1(0,b)$ bo'lsa, u holda ushbu

$$y(t)=f(t)+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds,\ \ t\in(0,b)$$

integral tenglama quyidagi yechimga ega bo'ladi:

$$y(t)=f(t)+\lambda\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left[\lambda(t-s)^{\alpha}\right]f(s)ds,\ \ t\in(0,b)\ .$$

Qo'sh tartibli Hilfer kasr tartibli hosilani izohda keltirilgan yozilish ifodasiga ko'ra (1) tenglamani quyidagicha yozib olamiz:

$$I_{0+}^{\gamma-\delta}D_{0+}^{\gamma}u(t)=\lambda u(t)+f(t).$$

Keyin, unga I_{0+}^{δ} integral operatorni tatbiq etamiz:

$$I_{0+}^{\delta}I_{0+}^{\gamma-\delta}D_{0+}^{\gamma}u(t)=\lambda I_{0+}^{\delta}u(t)+I_{0+}^{\delta}f(t),$$

$$I_{0+}^{\gamma}D_{0+}^{\gamma}u(t)=\lambda I_{0+}^{\delta}u(t)+I_{0+}^{\delta}f(t),$$

bu yerda $\gamma=\beta+\mu(n-\beta)$, $\delta=\beta+\mu(\alpha-\beta)$.

Riman-Liuvil kasr tartibli integral va differensial operatorlarning o'zaro kompozitsiyasi, ya'ni $I_{0+}^{\gamma}D_{0+}^{\gamma}$ ning ifodasidan foydalansak, quyidagiga kelamiz:

$$u(t)=\lambda I_{0+}^{\delta}u(t)+I_{0+}^{\delta}f(t)+\sum_{k=1}^{n}\frac{u_{n-\gamma}^{n-k}(0+)}{\Gamma(\gamma-k+1)}t^{\gamma-k},$$

yoki boshlang'ich shartlarni hisobga olsak, lemmada keltirilgan integral tenglamani hosil qilamiz:

$$u(t)=\lambda I_{0+}^{\delta}u(t)+h(t),$$

bunda

$$h(t)=I_{0+}^{\delta}f(t)+\sum_{k=1}^{n}\frac{\psi_{n-k}}{\Gamma(\gamma-k+1)}t^{\gamma-k}.$$

Lemmaga asosan bu integral tenglama quyidagi

$$u(t)=(t)+\lambda\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left[\lambda(t-s)^{\alpha}\right]h(s)ds=u_1+u_2$$

yechimga ega, ya'ni

$$u_1=\sum_{k=1}^{n}\frac{\psi_{n-k}}{\Gamma(\gamma-k+1)}t^{\gamma-k}+\lambda\int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left[\lambda(t-s)^{\delta}\right]\sum_{k=1}^{n}\frac{\psi_{n-k}}{\Gamma(\gamma-k+1)}s^{\gamma-k}ds=$$

$$=\sum_{k=1}^{n}\psi_{n-k}t^{\gamma-k}E_{\delta,\gamma-k+1}\left(\lambda t^{\delta}\right),$$

$$u_2 = I_{0+}^{\delta} f(t) + \lambda \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left[\lambda (t-s)^{\alpha}\right] I_{0+}^{\delta} f(s) ds =$$

$$= \int_0^t (t-s)^{\delta-1} E_{\delta,\delta}\left[\lambda (t-s)^{\delta}\right] f(s) ds \; .$$

u_1 va u_2 ifodalarni soddalashtirgandan so'ng, (1), (3) masalaning yechimini quyidagicha aniqlaymiz:

$$u(t) = \sum_{k=1}^{n} \psi_{n-k} t^{\gamma-k} E_{\delta,\gamma-k+1}\left(\lambda t^{\delta}\right) + \int_0^t (t-s)^{\delta-1} E_{\delta,\delta}\left[\lambda (t-s)^{\delta}\right] f(s) ds \; .$$

Ushbu yechimdan xususiy xosilali differensial tenglamalarni tadqiq qilishda foydalanish mumkin.

Eslatib o'tamizki, (1), (3) masala dastlab $0 < \alpha, \ \beta \le 1, 0 \le \mu \le 1$ bo'lgan hol uchun M. Bulavatsky tomonidan [3] da Laplas almashtirishi yordamida tadqiq qilingan.

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